Spectrum of rotating black holes and its implications for Hawking radiation

Saurya Das

Dept. of Physics, University of Lethbridge,
4401 University Drive, Lethbridge,
Alberta T1K 3M4, CANADA[*]

Himan Mukhopadhyay and P. Ramadevi

Dept. of Physics, Indian Institute of Technology, Bombay
Mumbai - 400 076, INDIA[†]

The reduced phase space formalism for quantising black holes has recently been extended to find the area and angular momentum spectra of four dimensional Kerr black holes. We extend this further to rotating black holes in all spacetime dimensions and show that although as in four dimensions the spectrum is discrete, it is not equispaced in general. As a result, Hawking radiation spectra from these black holes are continuous, as opposed to the discrete spectrum predicted for four dimensional black holes.

I. INTRODUCTION

Several different approaches have shown by now that the quantum mechanical spectra of black hole observables (such as area, charge and angular momentum) are discrete [1–3]. Among them, the reduced phase space quantisation technique of Barvinsky et al [2] confirms an earlier conjecture by Bekenstein and collaborators that the area spectrum of quantum black holes is indeed discrete as well as *equispaced* [4]. The last property makes a rather interesting prediction that Hawking radiation from black holes will have a noticeably discrete spectrum, which can have interesting experimental consequences [5]. The results were extended to the case of charged black holes, which yielded a separate discrete spectrum for the electric charge. The area spectrum was slightly more complicated in this case, although the uniform spacing property remained intact. These spectra were re-derived from algebraic approaches in [6, 7].

Recently the above formalism was extended by Gour and Medved to rotating (uncharged) black holes in four spacetime dimensions [8] (and to rotating charged black holes in [9]). They showed that the corresponding area spectrum is also discrete and equispaced: $A = 8\pi \left(n + m + 1/2 \right)$ where n and m are non-negative integers. While n signifies the departure of the black hole from extremality, m measures the classical angular momentum of the black hole. In this paper, we extend their formalism to include 3-dimensional BTZ black holes as well as 5 and higher dimensional Myers-Perry type rotating black holes with multiple angular momenta parameters. We show that while the area spectrum is discrete in each case, in general it is not equispaced. In particular, the 3 and 5 dimensional black holes have a non-uniform area spectrum. We also show that this makes Hawking radiation from these black holes continuous, as opposed to the discrete radiation spectrum characteristic of black holes of uniformly spaced area.

This paper is organised as follows. In the next section, we review some thermodynamic properties of the BTZ black hole and derive the quantum mechanical spectra of its area and angular momentum. In section (III), we consider the 5-dimensional rotating black hole with 2 angular momentum parameters and derive the corresponding spectra. Then in section (IV), we derive the area spectrum for six and higher dimensional black holes, whose area spectum is shown to be quite distinct from black holes in lower dimensions. In section (V), we study the implications of the area spectrum that we found for Hawking radiation, and show that in most cases the radiation spectrum is a continuum, *unlike* that for four dimensional black holes. We conclude with a summary, some open questions as well as some observations regarding adiabaticc invariants in quantum gravity in section (VI).

II. BTZ BLACK HOLES

First, let us consider a BTZ black hole, which is a solution of three dimensional Einstein equations with a negative cosmological constant $\Lambda \equiv -1/\ell^2$. Its horizon radii, area, entropy, Hawking temperature, angular velocity and the

---

first law of black hole mechanics that it satisfies are given by [10]

$$r_{\pm} = \ell \left[\frac{M}{2} \left(1 \pm \sqrt{1 - \left(\frac{J}{M\ell}\right)^2} \right) \right]^{1/2} \tag{1}$$

$$A = 2\pi r_+ \tag{2}$$

$$S_{BH} = \frac{A}{4G_3} = 4\pi r_+ \tag{3}$$

$$T_H = \frac{\kappa}{2\pi} = \frac{r_+^2 - r_-^2}{2\pi \ell^2 r_+} \tag{4}$$

$$\Omega = \frac{J}{2r_+^2} \tag{5}$$

$$dM = T_H \, dS_{BH} + \Omega \, dJ \ , \tag{6}$$

where κ is the surface gravity at the horizon and following [10] we have assumed that the three dimensional Newton's constant $G_3 = 1/8$. In the extremal limit $r_+ = r_-$ one has:

$$M\ell = J \ , \ \ (r_+)_{extr} = \ell\sqrt{\frac{M}{2}} = \sqrt{\frac{J}{2\ell}} \ , \ \ A_{extr} = \pi\sqrt{\frac{2J}{\ell}} \ , \ \ S_{extr} = 2\pi\sqrt{\frac{2J}{\ell}} \ , \tag{7}$$

where we have assumed $J \geq 0$ without loss of generality.

To derive quantum mechanical spectra, we start with the complete set of commuting observables (M, J) and their conjugates (Π_M, Π_J). Since Π_M has the interpretation of difference between Schwarzschild times across a spacelike slice extending from the left to the right wedge of a Kruskal diagram, following [2, 8], we impose the periodicity condition:

$$\Pi_M \sim \Pi_M + \frac{2\pi}{\kappa} \ , \tag{8}$$

to incorporate the thermodynamic nature of the system under consideration. Next, we make the following transformation to a new set of phase space variables (X, P_X) which automatically incorporate the above periodicity:

$$X = \sqrt{\frac{B(M, J)}{\pi}} \cos\left(\Pi_M \kappa\right) \tag{9}$$

$$P_X = \sqrt{\frac{B(M, J)}{\pi}} \sin\left(\Pi_M \kappa\right) \ , \tag{10}$$

where $B(M, J)$ is an hitherto unknown function. Computation of Poisson bracket yields:

$$\{X, P_X\} = \frac{\kappa}{2\pi} \frac{\partial B}{\partial M} \tag{11}$$

From (6) it follows that if the function $B(M, J)$ is chosen such that:

$$B(M, J) = S(M, J) + F(J) = 2A(M, J) + F(J) \ , \tag{12}$$

where $F(J)$ is arbitrary, then the above Poisson bracket becomes unity and the transformation $(M, \Pi_M) \to (X, \Pi_X)$ is indeed canonical. With this choice, squaring and adding (9) and (10), we get:

$$X^2 + P_X^2 = \frac{2A(M, J) + F(J)}{\pi} \ . \tag{13}$$

Noting that for a given J, since the horizon area A cannot be less than A_{extr} given in (7), it follows that $\forall \, J \geq 0$, the (X, P_X) phase space has a 'hole' centred at the origin and with radius Min $(2A(M, J) + F(J)) = 2A_{extr}(J) + F(J) = 2\pi\sqrt{2J/2\ell} + F(J)$ removed from it. Although this gives rise to quantisation ambiguities, these can be eliminated by the following unique choice of F:

$$F(J) = -2\pi\sqrt{\frac{2J}{\ell}} \ , \tag{14}$$

such that:

$$B(M, J) = 2A(M, J) - 2\pi\sqrt{\frac{2J}{\ell}} \qquad (15)$$

and

$$X^2 + P_X^2 = \frac{2A(M, J)}{\pi} - 2\sqrt{\frac{2J}{\ell}} \ . \qquad (16)$$

Quantisation is now straightforward with the replacements: $X \to \hat{X}$ and $P_X \to \hat{P}_X = -i\partial/\partial X$. The LHS of (16) is recognisable as the Hamiltonian of a harmonic oscillator of mass$= 1/2$ and angular frequency $= 2$. This results in the following spectra for entropy and horizon area:

$$A - \pi\sqrt{\frac{2J}{\ell}} = \pi\left(n + \frac{1}{2}\right) \ . \qquad (17)$$

Next, the quantisation of angular momentum follows from the usual identification in 3-dimensions: $\hat{J} = -i\partial/\partial\phi$, resulting in eigenfunctions: $\psi_m = \exp(im\phi)$ with eigenvalues $m = 0, 1, 2, \ldots$. Thus the final spectrum of area is:

$$A = \pi\left(n + \sqrt{\frac{2m}{\ell}} + \frac{1}{2}\right) \ , \quad n, m = 0, 1, 2, \ldots \qquad (18)$$

where it is understood that the area above (and in subsequent sections) is measured in Planck units. Note that it is discrete, but *not* equispaced. It will be shown in section (V) that this will have important consequences for Hawking radiation. Also note that $m = 0$ (no rotation) reduces the above spectrum to the equispaced spectrum conjectured in [4]. However, unlike the non-rotating case, or those that result from the recent identification of black hole quasi-normal modes with Hawking radiation frequencies (see e.g. [3, 11]), Eq.(18) does not have the multiplicative factor of $\ln(2)$ or $\ln(3)$ in front. Moreover, our 'ground-state area' is non-vanishing, suggesting that the the end stage of Hawking radiation is a Planck sized remnant. The above two features are generic for black holes in arbitrary spacetime dimensions, as will be seen in subsequent sections.

III. FIVE-DIMENSIONAL ROTATING BLACK HOLES

We now extend our results to five-dimensional rotating black holes. As suggested from the approaches in [2, 8] and in the previous section, the first step towards quantizing black hole horizon area is to find a set of mutually commuting operators and their conjugates. A naive extension of the analysis in the preceeding section fails for $d \geq 4$, since the different components of angular momenta fail to commute. However, as shown in [8] this problem can be readily cured by replacing the usual angular momentum components by the Euler components of angular momentum, which mutually commute [12]. Here we extend that analysis to $d = 5$ black holes which has *two* rotation parameters [13]. The key observation that makes this possible is that the relevant rotation group $SO(4)$ is isomorphic to $SU(2) \times SU(2)$. Defining the rotation and Lorentz generators in 5-spacetime dimensions:

$$\hat{L}_i = \frac{1}{2}\epsilon_{ijk}\hat{M}_{jk}, \qquad \hat{K}_i = \hat{M}_{i4} \qquad i, j, k = 1, 2, 3 \qquad (19)$$

where,

$$\hat{M}_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu) \qquad \mu, \nu = 1, 2, 3, 4 \ , \qquad (20)$$

their commutation relations follow:

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k, \qquad [\hat{L}_i, \hat{K}_j] = i\epsilon_{ijk}\hat{K}_k, \qquad [\hat{K}_i, \hat{K}_j] = i\epsilon_{ijk}\hat{L}_k \ . \qquad (21)$$

Next, defining two new angular momentum operators $\hat{J}_1$ and $\hat{J}_2$ as

$$\hat{J}_{1i} = \frac{1}{2}(\hat{L}_i + \hat{K}_i) \ , \qquad (22)$$

$$\hat{J}_{2i} = \frac{1}{2}(\hat{L}_i - \hat{K}_i) \ , \qquad (23)$$

it can be shown that

$$[\hat{J}_{1i}, \hat{J}_{1j}] = i\epsilon_{ijk}\hat{J}_{1k}, \qquad [\hat{J}_{2i}, \hat{J}_{2j}] = i\epsilon_{ijk}\hat{J}_{2k}, \qquad [\hat{J}_{1i}, \hat{J}_{2j}] = 0 \ . \tag{24}$$

giving two copies of the $SU(2)$ algebra. Accordingly, defining two copies of the analogs of Euler components of the angular momenta, $J_{\alpha i}, J_{\beta i}, J_{\gamma i}, (i = 1, 2)$, as follows:

$$J_{11} = -\cos\alpha_1 \cot\beta_1 J_{\alpha_1} - \sin\alpha_1 J_{\beta_1} + \frac{\cos\alpha_1}{\sin\beta_1}J_{\gamma_1} \ , \tag{25}$$

$$J_{12} = -\sin\alpha_1 \cot\beta_1 J_{\alpha_1} + \cos\alpha_1 J_{\beta_1} + \frac{\sin\alpha_1}{\sin\beta_1}J_{\gamma_1} \ , \tag{26}$$

$$J_{13} = J_{\alpha_1} \ , \tag{27}$$

$$J_{21} = -\cos\alpha_2 \cot\beta_2 J_{\alpha_2} - \sin\alpha_2 J_{\beta_2} + \frac{\cos\alpha_2}{\sin\beta_2}J_{\gamma_2} \ , \tag{28}$$

$$J_{22} = -\sin\alpha_2 \cot\beta_2 J_{\alpha_2} + \cos\alpha_2 J_{\beta_2} + \frac{\sin\alpha_2}{\sin\beta_2}J_{\gamma_2} \ , \tag{29}$$

$$J_{23} = J_{\alpha_2} \ , \tag{30}$$

where the six coordinates $\alpha_i, \beta_i, \gamma_i, \ i = 1, 2$ are 5-dimensional analogues of Euler angles in four dimensions. Finally, we define the two 'classical angular momenta':

$$J_{1Cl}^2 = J_{11}^2 + J_{12}^2 + J_{13}^2 = \frac{1}{\sin^2\beta_1}[J_{\alpha_1}^2 + J_{\gamma_1}^2 - 2\cos\beta_1 J_{\alpha_1}J_{\gamma_1}] + J_{\beta_1}^2 \tag{31}$$

$$J_{2Cl}^2 = J_{21}^2 + J_{22}^2 + J_{23}^2 = \frac{1}{\sin^2\beta_2}[J_{\alpha_2}^2 + J_{\gamma_2}^2 - 2\cos\beta_2 J_{\alpha_2}J_{\gamma_2}] + J_{\beta_2}^2. \tag{32}$$

Now the above two angular momenta are related to the ones that enter the five dimensional black hole metric in the following way [14]:

$$J_\phi = \frac{1}{2}(J_{1Cl} + J_{2Cl}) \tag{33}$$

$$J_\psi = \frac{1}{2}(J_{1Cl} - J_{2Cl}) \tag{34}$$

which in turn defines the parameters a and b as:

$$J_\phi = \frac{2}{3}Ma \tag{35}$$

$$J_\psi = \frac{2}{3}Mb \ , \tag{36}$$

implying:

$$J_{1Cl} = \frac{2}{3}M(a+b) \tag{37}$$

$$J_{2Cl} = \frac{2}{3}M(a-b) \ . \tag{38}$$

The two horizons are roots of the equation

$$(r^2 + a^2)(r^2 + b^2) - \mu r^2 = 0 \ . \tag{39}$$

The resulting thermodynamic quantities are (with $M \equiv \frac{3\pi}{8}\mu$):

$$2r_\pm^2 = \mu - a^2 - b^2 \pm \sqrt{(\mu - a^2 - b^2)^2 - 4a^2b^2} \tag{40}$$

$$A = \frac{2\pi^2\mu}{\kappa}\left[1 - \frac{a^2}{r_+^2 + a^2} - \frac{b^2}{r_+^2 + b^2}\right] \tag{41}$$

$$S_{BH} = \frac{A}{4G_5} \tag{42}$$

$$T_H = \frac{\kappa}{2\pi} = \frac{2r_+^2 + a^2 + b^2 - \mu}{2\pi\mu r_+} \tag{43}$$

$$\Omega_\phi = \frac{a}{r_+^2 + a^2} \tag{44}$$

$$\Omega_\psi = \frac{a}{r_+^2 + b^2} \tag{45}$$

$$dM = T_H \, dS_{BH} + \Omega_\phi \, dJ_\phi + \Omega_\psi \, dJ_\psi \ , \tag{46}$$

$$\tag{47}$$

From (40) the extremality bound follows:

$$\mu \geq a^2 + b^2 + 2|ab| \qquad \text{or equivalently,}$$

$$M^3 \geq \frac{27\pi}{32}(J_\phi^2 + J_\psi^2 + 2|J_\phi J_\psi|) \ . \tag{48}$$

From (40), (41) and (43) it follows that:

$$A = \frac{2\pi^2\mu^2 r_+}{2r_+^2 + a^2 + b^2 - \mu} \frac{r_+^4 - (ab)^2}{(r_+^2 + a^2)(r_+^2 + b^2)} \ , \tag{49}$$

$$T_H = \frac{\sqrt{(\mu - a^2 - b^2)^2 - (2ab)^2}}{2\pi\mu r_+} \ , \tag{50}$$

which along with Eq.(40) implies the following conditions in the extremal limit $(T_H = 0)$:

$$\mu = (a+b)^2 \tag{51}$$

$$r_{ext}^2 = |ab| = \frac{9}{4M^2}|J_\phi J_\psi| \ . \tag{52}$$

Thus, from (49) it follows that that the horizon area in the extremal limit is:

$$A_{ext} = 8\pi\sqrt{J_\phi J_\psi} = 4\pi\sqrt{|J_{1Cl}^2 - J_{2Cl}^2|} = A_{ext}(J_{1Cl}, J_{2Cl}) \ . \tag{53}$$

Now, it can be seen from (31)and (32) that J_{iCl}^2 commutes with $J_{\alpha i}$ and $J_{\gamma i}$, for $i = 1, 2$. Thus, one can choose as canonical variables the following set: $(M, J_{1Cl}, J_{\alpha_1}, J_{\gamma_1}, J_{2Cl}, J_{\alpha_2}, J_{\gamma_2})$ with the corresponding conjugates $(\Pi_M, \Pi_{J_1}, \Pi_{\alpha_1}, \Pi_{\gamma_1}, \Pi_{J_2}, \Pi_{\alpha_2}, \Pi_{\gamma_2})$ respectively. As before, we impose the periodicity (8), which is now incorporated by the following transformation:

$$X = \sqrt{\frac{B(M, J_{1Cl}, J_{\alpha_1}, J_{\gamma_1}, J_{2Cl}, J_{\alpha_2}, J_{\gamma_2})}{\pi}}\cos(\Pi_M\kappa) \ , \tag{54}$$

$$P_X = \sqrt{\frac{B(M, J_{1Cl}, J_{\alpha_1}, J_{\gamma_1}, J_{2Cl}, J_{\alpha_2}, J_{\gamma_2})}{\pi}}\sin(\Pi_M\kappa) \ . \tag{55}$$

Computation of Poisson brackets yields once again:

$$\{X, P_X\} = \frac{\kappa}{2\pi}\frac{\partial B}{\partial M} \ , \tag{56}$$

which now implies:

$$B = \frac{1}{4}[A(M, J_{1Cl}, J_{2Cl}) - A_{ext}(J_{1Cl}, J_{2Cl})] \ , \tag{57}$$

where we have set $G_5 = 1$. Squaring and adding (54) and (55):

$$X^2 + P_X^2 = \frac{1}{4\pi}[A(M, J_{1Cl}, J_{2Cl}) - A_{ext}(J_{1Cl}, J_{2Cl})] \geq 0 \ . \tag{58}$$

Quantising as before, we now get:

$$A - A_{ext} = 8\pi\left(n + \frac{1}{2}\right) \ , \quad n = 0, 1, 2, \cdots \tag{59}$$

It may be noted that the above equation has the same form as the corresponding equation in the $d = 4$ case (Eq.(43) of [8]). The distinction between that case and the current one lies in the difference in the functional form of extremal entropy and the resulting difference that arises on quantisation. Now, since (25)-(30) remains valid when the classical quantities are replaced by their corresponding operator counterparts [8, 12], together with the replacement: $\hat{J}_{ai} \to -i\frac{\partial}{\partial a_i}$, $a = \alpha, \beta, \gamma$, one obtains the following relations for $i = 1, 2$:

$$\hat{J}_{iCl}^2 = \frac{1}{\sin^2 \beta_i}[\hat{J}_{\alpha_i}^2 + \hat{J}_{\gamma_i}^2 - 2\cos\beta_i\hat{J}_{\alpha_i}\hat{J}_{\gamma_i}] + \hat{J}_{\beta_i}^2 , \tag{60}$$

$$\hat{J}_i^2 = \hat{J}_{i1}^2 + \hat{J}_{i2}^2 + \hat{J}_{i3}^2 . \tag{61}$$

$$\text{and} \quad \hat{J}_{iCl}^2 - \hat{J}_i^2 = \cot\beta_i\frac{\partial}{\partial\beta_i} . \tag{62}$$

Since all the operators in the set $\{\hat{J}_1, \hat{J}_2, \hat{J}_{1Cl} \text{ and } \hat{J}_{2Cl}\}$ commute with $\hat{J}_{\alpha_i}$ and $\hat{J}_{\gamma_i}$ ($i = 1, 2$), both sets of eigenvectors $\{|j_1, j_2, J_{\alpha_1}, J_{\alpha_2}, J_{\gamma_1}, J_{\gamma_2}\rangle\}$ and $\{|J_{1Cl}, J_{2Cl}, J_{\alpha_1}, J_{\alpha_2}, J_{\gamma_1}, J_{\gamma_2}\rangle\}$ are complete and an element of one can be expressed as a superposition of the elements of the other:

$$|J_{1Cl}, J_{2Cl}, J_{\alpha_1}, J_{\alpha_2}, J_{\gamma_1}, J_{\gamma_2}\rangle = \sum_{j_1, j_2} C_{j_1, j_2, J_{1Cl}, J_{2Cl}}(j_1, j_2, J_{1Cl}, J_{2Cl})|j_1, j_2, J_{\alpha_1}, J_{\alpha_2}, J_{\gamma_1}, J_{\gamma_2}\rangle , \tag{63}$$

To compute the eigenvalues of J_{iCl}, consider the eigenfunction with zero eigenvalues for α_i and γ_i: $\Psi_{J_{1Cl},0,0,0,0,0} = \langle\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2|J_{1Cl}, 0, 0, 0, 0, 0\rangle$. Then, from (60):

$$\hat{J}_{iCl}^2\Psi_{J_{iCl},0,0,0,0,0} = \hat{J}_{\beta_i}^2\Psi_{J_{iCl},0,0,0,0,0} = -\frac{\partial^2}{\partial\beta_1^2}\Psi_{J_{iCl},0,0,0,0,0} \equiv m_i^2\ \Psi_{J_{iCl},0,0,0,0,0} . \tag{64}$$

With the usual identification for Euler angles, $\beta_i + \pi = \pi - \beta_i$, we get:

$$\Psi_{J_{iCl},0,0,0,0,0} \sim \cos(m_i\beta_i) , \quad m_i = 0, 1, 2. \cdots$$
$$\sim \sin(m_i\beta_i) , \quad m_i = 1/2, 3/2, 5/2, \cdots \tag{65}$$
$$\text{with} \quad J_{iCl} = m_i . \tag{66}$$

The additional fact that $\Psi_{j_i,0,0,0,0}$ is a symmetric function of β together with equation (63) rules out the half-odd integral quantisation of J_{iCl}. Thus we are left with Eq(65) as the correct quantisation condition. This implies, from (33) and (34) that:

$$J_\phi = \frac{1}{2}(m_1 + m_2) \tag{67}$$

$$J_\psi = \frac{1}{2}(m_1 - m_2) , \quad m_i = 0, 1, 2. \cdots \tag{68}$$

Using (53), the final result of the area spectrum is

$$A = 8\pi\left(n + \frac{1}{2}\sqrt{|m_1^2 - m_2^2|} + \frac{1}{2}\right) , \quad n, m_1, m_2 = 0, 1, 2, \cdots \tag{69}$$

Thus we see that the five-dimensional Kerr black hole area spectrum is also quantized but not equally spaced. Equispaced spectrum can arise as a special case however, either when $m_1 = 0$ (or $m_2 = 0$) or when $m_1 = m_2$. The latter case implies $J_\psi = 0$, corresponding to a single parameter five dimensional black hole. We will examine in section (V) the effects of quantized area spectrum on Hawking radiation.

IV. SIX AND HIGHER DIMENSIONAL ROTATING BLACK HOLES

The situation simplifies drastically for six and higher dimensions because of the absence of an extremality bound. The number of angular momentum parameters a_i's for the d-dimensional Kerr black hole will be $[(d-1)/2]$(square bracket denotes integer part). For simplicity, we consider the metric component of a single angular momentum parameter Kerr black hole- that is, $a_1 \equiv a \equiv 2J/M$ where $M = \mu(d-2)(2\pi)^{(d-1)/2}/\{\Gamma[(d-1)/2]16\pi G\}$ and $a_i = 0$ for $2 \le i \le [(d-1)/2]$ [13]:

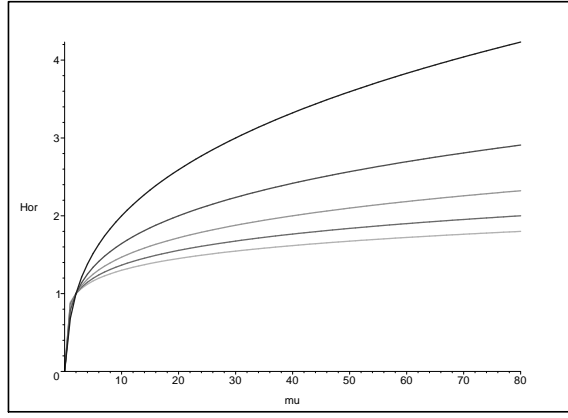$$g^{rr} = \frac{r^{d-5}(r^2 + a^2) - \mu}{r^{d-5}(r^2 + a^2\cos^2\theta)} . \tag{70}$$

FIG. 1: Plot of r_+ vs μ for $d = 6, 7, \ldots, 10$ for $a = 1$. The highest graph corresponds to $d = 6$ and the lowest to $d = 10$.

It follows that for $d \geq 6$, the horizon condition $g^{rr} = 0$ is satisfied for some $0 \leq r \leq \infty$ for any given value of a and M. In other words, the black hole angular momentum is in no way contrained by its mass and in fact can be arbitrarily large. Moreover, for a fixed a, for $\mu \ll a$, we see that the horizon radius shrinks to zero. This can also be seen from Fig.1, where the outer horizon radius has been plotted against the black hole mass for $6 \leq d \leq 10$ Thus, there is no 'minimum' horizon area for a fixed a, and the function $F(J)$ can be chosen to vanish. Consequently, the equivalents of (54)-(55) read as:

$$X = \sqrt{\frac{A(M,J)}{4\pi}} \, \cos(\Pi_M \kappa) \,, \tag{71}$$

$$P_X = \sqrt{\frac{A(M,J)}{4\pi}} \, \sin(\Pi_M \kappa) \,. \tag{72}$$

where

$$\kappa = \frac{\partial \Pi/\partial r - 2\mu r}{2\mu r^2}\Big|_{r_+} \,, \quad d = \text{odd} \tag{73}$$

$$= \frac{\partial \Pi/\partial r - \mu}{2\mu r}\Big|_{r_+} \,, \quad d = \text{even} \tag{74}$$

and $\Pi = \prod_{i=1}^{\lceil \frac{d-1}{2} \rceil} \left(r^2 + a_i^2 \right)$. Once again (71)-(72) incorporate the natural periodicity condition (8) and the transformations are canonical. Squaring and adding:

$$A = 4\pi \left(X^2 + P_X^2 \right) \,. \tag{75}$$

Quantisation yields:

$$A_n = 8\pi \left(n + \frac{1}{2} \right) \,, \quad n = 0, 1, 2, \cdots. \tag{76}$$

Remarkably, angular momentum plays no role in area spectrum in $d \geq 6$, and the above spectrum is *identical* to the neutral black hole spectrum found in [2]. The Hawking radiation spectrum is not the same however, as will be shown in the next section. It would be interesting to investigate other physical significances that this equispaced spectrum might have.

V.   HAWKING RADIATION FROM ROTATING BLACK HOLES

An important consequence of a strictly equispaced area spectrum is a markedly discrete spectrum for Hawking radiation [2, 4, 5]. Consider for example the area spectrum of a $d = 4$ Schwarzschild black hole. Using $A = 4\pi r_+^2 =$

$16\pi M^2$, the lowest Hawking radiation frequency ω_0 emitted due to a transition of this black hole from an excited state to its next lower state ($\delta n = -1$, $|\delta A| = 8\pi$ from (76)) is given by:

$$\omega_0 = |\delta M| = \frac{|\delta A|}{32\pi M} = \frac{1}{4M} = 2\pi T_H \tag{77}$$

and all higher frequencies will be multiples of the above. Comparing with Wien's law for frequency of maximum intensity of blackbody radiation:

$$\omega_{max} \approx T_H \ , \tag{78}$$

we see that the $\omega_0 \approx \omega_{max}$. Thus there will be only a few visible spectrum lines in Hawking radiation *even for a macroscopic black hole*, leading to the aforementioned conclusion. The qualitative picture remains unchanged for Reissner-Nordström black holes as well [2]. Next we examine the impications for rotating black holes.

A. BTZ black hole

In this case, it follows from (18) that for an emitted quantum of Hawking radiation:

$$\delta S_{BH} = 2\delta A = 2\pi \left[\delta n + \frac{\delta m}{\sqrt{2\ell m}} \right] \ . \tag{79}$$

Replacing $dM \to \omega_0$ and plugging in the expression for Ω from (5) in the first law (6), we get:

$$\omega_0 = 2\pi T_H \ \delta n + \left[\frac{2\pi T_H}{\sqrt{2\ell m}} + \frac{m}{2\left(n + \sqrt{\frac{2m}{\ell}} + 1/2\right)^2} \right] \delta m \tag{80}$$

Next, consider an emission line for which $\delta n = 0$ and $|\delta m| = 1$, and the black hole to be 'macroscopic', for which $n \geq m \gg 1$. Then:

$$\omega_0 = \frac{2\pi T_H}{\sqrt{2\ell m}} + \frac{m}{2\left(n + \sqrt{\frac{2m}{\ell}} + 1/2\right)^2} \to 0 \tag{81}$$

implying that the radiation spectrum is quasi-continuum, quite different from the $d = 4$ case and exactly as predicted by the original Hawking analysis. We will return to the case $\delta n \neq 0$ for this and other black holes in the concluding section.

B. Four dimensional Kerr black hole

Here we use the spectrum found in [8], namely:

$$A = 8\pi \left(n + m + \frac{1}{2}\right) \ , \quad n, m = 0, 1, 2, \cdots , \tag{82}$$

$$J_{Cl} = m \tag{83}$$

implying

$$\delta A = 8\pi \left(\delta n + \delta m\right) \ , \quad \delta n, \delta m = 0, 1, 2, \cdots . \tag{84}$$

Once again, using (82)-(83), the first law and $\Omega = a/(r_+^2 + a^2)$, we now get:

$$\omega_0 = 2\pi T_H \ \delta n + \left[2\pi T_H + \frac{m}{\frac{M}{2}\left(n + m + \frac{1}{2}\right) + \frac{m^2}{M}} \right] \delta m \tag{85}$$

In this case, for $|\delta m| = 1$ (and for any finite δn), we see that:

$$\omega_0 \approx 2\pi T_H \tag{86}$$

and the Hawking radiation spectrum is once again discrete and identical to the Schwarzschild (and Reissner-Nordström) black hole spectrum (77).

C. Five dimensional rotating black hole

From the five dimensional area spectrum (69), the first law and the expression for the angular momenta (44-45), we get:

$$\omega_0 = 2\pi T_H \left(|\delta n| + \frac{m_1|\delta m_1| - m_2|\delta m_2|}{2\sqrt{m_1^2 - m_2^2}} \right) + \frac{3}{8M} \left[\frac{(m_1 + m_2)d(m_1 + m_2)}{r_+^2 + 9(m_1 + m_2)^2/16M^2} + (m_2 \to -m_2) \right] \tag{87}$$

where we have assumed without loss of generality that $m_1 \geq m_2$. As for BTZ, consider an emission characterised by: $\delta n = 0$ (signifying $A - A_{ext}$ remaining unchanged) and concentrate on the second term within the parenthesis, since the last term is in any case negligible for very large M. We see that for any given $\epsilon \ll 1$, if

$$\delta m_1 = \left(\frac{m_2}{m_1} \right) \left[1 + \frac{2\epsilon}{\delta m_2} \sqrt{\left(\frac{m_1}{m_2} \right)^2 - 1} \right] \delta m_2 \approx \left(\frac{m_2}{m_1} \right) \delta m_2 , \tag{88}$$

then

$$\omega_0 = 2\pi\epsilon \, T_H \to 0 \tag{89}$$

Notice however that both $|\delta m_{1,2}| \geq 1$. This implies, along with (88) and $|\delta m_2| \leq m_2$, that $m_2 \leq m_1 \leq m_2^2$. That is, unless the two angular momentum parameters are hugely disproportionate, such that the second inequality in the above is violated (or when $m_1 = 0$ or $m_2 = 0$), the radiation spectrum is a quasi-continuum, as was the case in three dimensions. Since these black holes are natural candidates in certain brane world scenarios, it is possible that the above observations will have experimental signatures from the four dimensional (brane) point of view.

D. Six and higher dimensional rotating black hole

From (76) and the first law, we get:

$$\omega_0 = 2\pi T_H \, \delta n + \frac{a}{r_+^2 + a^2} \, \delta m . \tag{90}$$

Note that, as for $d = 4$ Schwarzschild black holes, the first term implies a discrete spectrum. But unlike that case, here an additional parameter δm $(\delta m_i, i = 1, 2, [d-1]/2$ in general) enters the problem. Recalling that there is no extremality bound for $d \geq 6$, if the black hole is such that $a \gg r_+$ and the black hole to be macroscopic, then for $\delta n = 0$, $|\delta m| = 1$, we have:

$$\omega_0 = \frac{1}{a} \to 0. \tag{91}$$

It is easy to see that similar conclusions will follow for $a \ll r_+$ as well as $a \approx r_+$. Thus we see that the single angular momentum Kerr black holes in $d \geq 6$ shows continuous Hawking spectrum in spite of the area spectrum being equispaced. It remains to be checked whether this property holds even if we include more angular momentum parameters.

VI. CONCLUSIONS

In this paper, we have extended an earlier formalism by Barvinsky et al, adapted to Kerr black holes by Gour and Medved to include rotating black holes in all spacetime dimensions. This was done in three steps. First, the method was applied to 3-dimensional BTZ black holes with one angular momentum parameter, where the observables A and J were mutually commuting. Second, it was applied to 5-dimensional rotating black holes with two angular momentum parameters. Using the fact that the corresponding rotation group $SO(4) \cong SU(2) \times SU(2)$, and using the technique of [8] in which the Cartesian components of angular momentum are replaced by the mutually commuting Euler components, we were able to quantise the the two angular momentum components and arrive at the spectrum of horizon area of these black holes. Finally six and higher dimensional black holes were studied with one angular momentum parameter, since this was sufficient to demonstrate its essential difference with lower dimensional black holes,

The horizon area spectra for the above black holes were found to be quite different from that in $d = 4$. In particular, we found that both for $d = 3$ and 5 the spectrum is no longer equispaced (except for the special case when the two angular momentum parameters were equal in $d = 5$). A direct consequence was that Hawking radiation spectrum from these black holes are practically continuous (for large black holes), as opposed to the distinctly discrete spectrum predicted for $d = 4$ black holes which can carry angular momentum or charge. The situation was even more interesting for $d = 6$, in which case although the spectrum turned out to be equispaced, the Hawking radiation retained its continuous nature. Although it may seem that there is an additional discrete spectrum for all the cases considered when $\delta n \approx 1$, this does not affect our conclusions in any way, since the superposition of a discrete and a continuous spectrum is a continuum, the greybody factors being the same for each case. Such closeness of radiation spectra from higher dimensional black holes with the semi-classical Hawking analysis may lead one to speculate that brane world black holes are more fundamental, since Hawking's result is expected to be valid for macroscopic black holes. In fact one of the proposed brane world scenarios requires the spacetime dimension of our universe to be at least six [15], for which we have seen that interesting conclusions follow for rotating black holes. However, further investigations are required before arriving at a definitive conclusion. In any case, phenomenological implications of our higher dimensional results seem worth investigating.

We conclude with an observation related to *adiabatic invariants* in quantum gravity. Motivated by some thought experiments, it was conjectured by Bekenstein that black hole horizon area is an adiabatic invariant in quantum gravity, i.e. it remains unaffected by small changes in black hole parameters. In [2] an explicit proof of this conjecture was presented for spherically symmetric uncharged black holes, while in the presence of a charge Q it was shown that $A - A_{extr}(Q)$ was an adiabatic invariant. We show here that a similar result follows. For periodic systems, it is well known that the quantity:

$$\mathcal{I} \equiv \oint P_X \, dX \tag{92}$$

is an adiabatic invariant, where (X, Π_X) are usual phase-space coordinates [16]. It follws that for harmonic oscillator like Hamiltonians as in (16), (58) and (75), the corresponding adiabatic invariants are:

$$
\begin{aligned}
\mathcal{I} &= \frac{A - A_{extr}(J)}{\pi} \quad (BTZ) \\
&= \frac{A - A_{extr}(J_{1Cl}, J_{2Cl})}{8\pi} \quad (d = 5) \\
&= \frac{A}{8\pi} \quad (d \geq 6) \ .
\end{aligned}
$$

Note once again that as for charged black holes, upto $d = 5$, the 'excess-area' above extremality is an adiabatic invariant, while for $d \geq 6$, the horizon area itself is the invariant, as conjectured by Bekenstein. It will be interesting to study the most general case of charged and rotating black holes to see what the adiabatic invariants are for such a system.

Acknowledgements:

---

[1] Ya. I. Kogan JETP Lett. **44** (1986) 267, Pizma Zh. Eksp. Teor. Fiz. **44** (1986) 209; V. A. Berezin, Phys. Rev. **D55** (1997) 2139 (gr-qc/9602020), Nucl. Phys. Proc. Suppl. **57** (1997) 181 (gr-qc/9701017); V. A. Berezin, A. M. Boyarsky, A. Yu. Neronov, Phys. Lett. **B455** (1999) 109 (gr-qc/9808027); C. Rovelli and L. Smolin, Nucl. Phys. **B442** (1995) 593 (gr-qc/9411005); A. Ashtekar and J. Lewandowski, Class. Quant. Grav. **14** (1997) A55 (gr-qc/9602046); H. Kastrup, Phys. Lett. **B385** (1996) 75 (gr-qc/9605038); J. Louko and J. Makela, Phys.Rev. **D54** (1996) 4982 (gr-qc/9605058); J. Makela and P. Repo, Phys.Rev. **D57** (1998) 4899 (gr-qc/9708029); C. Vaz and L. Witten, Phys. Rev. **D60** (1999) 024009 (gr-qc/9811062), Phys. Rev. **D63** (2001) 024008 (gr-qc/0006039); R. Garattini, Phys. Lett. **459** (1999) 461 (hep-th/9906074), Entropy **2** (2000) 26 (gr-qc/0003089), Int. J. Mod. Phys. **A17** (2002) 1965 (gr-qc/0111068); M. Bojowald, H.A. Kastrup, F. Schramm and T, Strobl, Phys. Rev. **D62** (2000) 044026 (gr-qc/9906105); A. Alekseev, A. P. Polychronakos, M. Smedbäck, Phys. Lett. **B574** (2003) 296 (hep-th/0004036); J.Makela, P. Repo, M. Luomajoki and J. Piilonen, Phys. Rev. **D64** (2001) 024018 (gr-qc/0012055).
[2] A. Barvinsky and G. Kunstatter, in Proceedings of the Second International A.D. Sakharov Conference on Physics, Eds. I. M. Dremin and A.M. Seminkhatov, (World Scientific, Singapore, 1997), pp.210-215. (gr-qc/9607030); A. Barvinsky, S. Das,

G. Kunstatter, Class. Quant. Grav. **18** (2001) 4845 (gr-qc/0012066) and Phys. Lett. **B517** (2001) 415 (hep-th/0102061); Found. Phys. **32** (2002) 1851 (hep-th/0209039).

[3] G. Kunstatter, Phys. Rev. Lett. **90** (2003) 161301 (gr-qc/0212014); E. Abdalla, K. H. C. Castello-Branco, A. Lima-Santos, Mod. Phys. Lett. **A18** (2003) 1435 (gr-qc/0301130); A. P. Polychronakos, Phys.Rev. **D69** (2004) 044010 (hep-th/0304135); M. R. Setare, Class. Quant. Grav. **21** (2004) 1453 (hep-th/0311221), Phys. Rev. **D69** (2004) 044016 (hep-th/0312061), hep-th/0401063; M. R. Setare, E. C. Vagenas, hep-th/0401187; J. Kettner, G. Kunstatter, A. J. M. Medved, Class. Quant. Grav. **21** (2004) 5317 (gr-qc/0408042).

[4] J. D. Bekenstein, Lett. Nuovo Cimento, **11** (1974) 467; V.Mukhanov, JETP Letters **44** (1986) 63; J. D. Bekenstein and V. F. Mukhanov, Phys. Lett. **B360** (1995) 7 (gr-qc/9505012).

[5] L. Smolin, Matters of Gravity **7** (1996) 10 (gr-qc/9602001); M. Barreira, M. Carfora, C. Rovelli, Gen. Rel. Grav. **28** (1996) 1293 (gr-qc/9603064).

[6] J. D. Bekenstein, Proceedings of Marcel Grossmann Meeting, Jerusalem (1997) (gr-qc/9710076); and references therein; J. D. Bekenstein, Proceedings of *2001: A Spacetime Odyssey*, World Scientific (hep-th/0107045); J. D. Bekenstein, in Advances in Interplay between Quantum and Gravity Physics, ed. P. G. Bergmann and V. de Sabbata (Kluwer, Dordrecht 2002), pp.1-26 (gr-qc/0107049); J. D. Bekenstein, G. Gour, Phys. Rev. **D66** (2002) 024005 (gr-qc/0202034).

[7] S. Das, P. Ramadevi, U. A. Yajnik, Mod. Phys. Lett. **A 17** (2002) 993) (hep-th/0202076); S. Das, P. Ramadevi, U. A. Yajnik, A. Sule, Phys. Lett. **B565** (2003) 201 (hep-th/0207169).

[8] G. Gour, A. J. M. Medved, Class. Quant. Grav. **20** (2003) 2261 (hep-th/0211089).

[9] G. Gour, A. J. M. Medved, Class. Quant. Grav. **20** (2003) 1661 (gr-qc/0212021).

[10] M. Banãdos, C. Teitelboim, J. Zanelli, Phys. Rev. Lett. **69** (1992) 1849 (hep-th/9204099).

[11] S. Hod, Phys. Rev. Lett. **81** (1998) 4293 (gr-qc/9812002); O. Dreyer, Phys. Rev. Lett. **90** (2003) 081301 (gr-qc/0211076); L. Motl, A. Neitzke, Adv. Theor. Math. Phys. **7** (2003) 307 (hep-th/0301173).

[12] L. D. Landau and E. M. Lifschitz, *Quantum Mechanics, Non-Relativistic Theory*, Third Edition, Pergamon Press, Oxford, 1977, p.416.

[13] R. C. Myers, M. J. Perry, Annals Phys. **172** (1986) 304.

[14] S. W. Hawking, C. J. Hunter, M. M. Taylor-Robinson, Phys. Rev. **D59** (1999) 64005 (hep-th/9811056).

[15] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **429**, 263 (1998) (hep-ph/9803315); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **436**, 257 (1998) (hep-ph/9804398); I. Antoniadis, Phys. Lett. B **246**, 377 (1990).

[16] L. D. Landau, E. M. Lifschitz, *Mechanics*, Pergamon Press (1976).